CROOKED HALFSPACES

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Dedicated to the memory of Robert Miner

Abstract. We develop the Lorentzian geometry of a crooked halfspace in $2+1$-dimensional Minkowski space. We calculate the affine, conformal and isometric automorphism groups of a crooked halfspace, and discuss its stratification into orbit types, giving an explicit slice for the action of the automorphism group. The set of parallelism classes of timelike lines, or *particles*, in a crooked halfspace is a geodesic halfplane in the hyperbolic plane. Every point in an open crooked halfspace lies on a particle. The correspondence between crooked halfspaces and halfplanes in hyperbolic 2-space preserves the partial order defined by inclusion, and the involution defined by complementarity. We find conditions for when a particle lies completely in a crooked half space. We revisit the disjointness criterion for crooked planes developed by Drumm and Goldman in terms of the semigroup of translations preserving a crooked halfspace. These ideas are then applied to describe foliations of Minkowski space by crooked planes.

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Contents

1. Introduction

Crooked planes are special surfaces in $2 + 1$-dimensional Minkowski space E. They were introduced by the third author [10] to construct fundamental polyhedra for nonsolvable discrete groups Γ of isometries which act properly on all of E. The existence of such groups Γ was discovered by Margulis [18, 19] around 1980 and was quite unexpected (see Milnor [20] for a lucid description of this problem and [15] for related results).

In this paper we explore the geometry of crooked planes and the polyhedra which they bound.

The basic object is a *(crooked) halfspace.* A *halfspace* is one of the two components of the complement of a crooked plane $\mathcal{C} \subset \mathsf{E}$. It is the interior of a 3-dimensional submanifold-with-boundary, and the boundary $\partial\mathcal{H}$ equals $\mathcal{C}$.

Every crooked halfspace $\mathcal{H}$ determines a *halfplane* $\mathfrak{h} \subset \mathsf{H}^2$, consisting of directions of timelike lines completely contained in $\mathcal{H}$. Two halfspaces determine the same halfplane if and only if they are *parallel,* that is, they differ by a translation. The translation is just the unique translation between the respective vertices of the halfspaces. We call $\mathfrak{h}$ the *linearization* of $\mathcal{H}$ and denote it $\mathfrak{h} = \mathsf{L}(\mathcal{H})$. The terminology is motivated by the fact that the linear holonomy of a complete flat Lorentz manifold defines a complete hyperbolic surface. See §5 for a detailed explanation. In our previous work ([5, 6, 7], we have used crooked planes to extend constructions in 2-dimensional hyperbolic geometry to Lorentzian 3-dimensional geometry.

The set $\mathfrak{S}(\mathsf{H}^2)$ of halfplanes in H^2 enjoys a partial ordering given by inclusion and an involution given by the operation of taking the complement. Similarly the set $\mathfrak{S}(\mathsf{E})$ of crooked halfspaces in E is a partially ordered set with involution.

Theorem. *Linearization* $\mathfrak{S}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathfrak{S}(\mathsf{H}^2)$ *preserves the partial relation defined by inclusion and the involution defined by complement.*

Furthermore, we show that any point in a crooked halfspace $\mathcal{H}$ lies on a particle determing a timelike direction in the halfplane $\mathsf{L}(\mathcal{H}) \subset \mathsf{H}^2$.

Crooked halfspaces enjoy a high degree of symmetry, which we exploit for the proofs of these results. In this paper we consider automorphisms preserving the Lorentzian structure up to isometry, the Lorentzian structure up to conformal equivalence, and the underlying affine connection.

Theorem. *Let* $\mathcal{H} \subset \mathsf{E}$ *be a crooked halfspace. Its respective groups of orientation-preserving affine, conformal and isometric automorphisms are:*

$$\begin{aligned}\mathsf{Aff}^+(\mathcal{H}) &\cong \mathbb{R}^3 \rtimes \mathbb{Z}/2\\ \mathsf{Conf}^+(\mathcal{H}) &\cong \mathbb{R}^2 \rtimes \mathbb{Z}/2\\ \mathsf{Isom}^+(\mathcal{H}) &\cong \mathbb{R}^1 \rtimes \mathbb{Z}/2\end{aligned}$$

The involutions preserving $\mathcal{H}$ *are reflections in tachyons orthogonal to the spine of* $\mathcal{H}$*, which preserve orientation on* E *but reverse time-orientation.*

A fundamental notion in crooked geometry is the *stem quadrant* $\mathsf{Quad}(\mathcal{H})$ of a halfspace $\mathcal{H}$, related the subsemigroup $\mathsf{V}(\mathcal{H})$ of V consisting of translations preserving $\mathcal{H}$. The disjointness results of [13] can be easily expressed in terms of this cone of translations. In particular we prove:

Theorem. *Two crooked halfspaces are disjoint if and only if*

$$\mathsf{Vertex}(\mathcal{H}_1) - \mathsf{Vertex}(\mathcal{H}_2) \in \mathsf{V}(\mathcal{H}_1) - \mathsf{V}(\mathcal{H}_2). \tag{1}$$

Finally these ideas are exploited to construct foliations of E by crooked planes. Following our basic theme, we begin with a geodesic foliation of H^2 and extend it to a crooked foliation of E. Such foliations may be useful in understanding the deformation theory and the geometry of Margulis spacetimes.

2. Lorentzian geometry

2.1. $(2+1)$**-dimensional Minkowski space.** A *Lorentzian vector space of dimension 3* is a real 3-dimensional vector space V endowed with an inner product of signature $(2,1)$. The Lorentzian inner product will be denoted:

$$\begin{aligned} \mathsf{V} \times \mathsf{V} &\longrightarrow \mathbb{R} \\ (\mathbf{v}, \mathbf{u}) &\longmapsto \mathbf{v} \cdot \mathbf{u} \end{aligned}$$

We also fix an orientation on V. The orientation determines a nondegenerate alternating trilinear form

$$\mathsf{V} \times \mathsf{V} \times \mathsf{V} \xrightarrow{\mathsf{Det}} \mathbb{R}$$

which takes a positively oriented orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with inner products

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \ \mathbf{e}_3 \cdot \mathbf{e}_3 = -1$$

to 1. The oriented Lorentzian 3-dimensional vector space determines an alternating bilinear mapping $\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$, called the *Lorentzian cross-product,* defined by

$$\mathsf{Det}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}. \tag{2}$$

Compare, for example, with [13].

Denote the group of orientation-preserving linear isometries of V by $\mathsf{SO}(2,1)$. The group of orientation-preserving linear *conformal automorphisms* of V is the product $\mathsf{SO}(2,1) \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the group of *positive homotheties* $\mathbf{v} \mapsto \lambda\mathbf{v}$, where $\lambda > 0$. Denote the group of orientation-preserving linear automorphisms of V by $\mathsf{GL}^+(3,\mathbb{R})$.

In this paper, *Minkowski space* E will mean a 3-dimensional oriented geodesically complete 1-connected flat Lorentzian manifold. It is naturally an affine space having as its group of translations an oriented 3-dimensional Lorentzian vector space V. Two points $p, q \in \mathsf{E}$ differ by a unique translation $\mathbf{v} \in \mathsf{V}$, that is, there is a unique vector $\mathbf{v}$ such that

$$\mathbf{v} := p - q \in \mathsf{V}.$$

We also write $p = q + \mathbf{v}$. Identify E with V by choosing a distinguished point $o \in \mathsf{V}$, which we call an *origin.* For any point $p \in \mathsf{E}$ there is a unique vector $\mathbf{v} \in \mathsf{V}$ such that $p = o + \mathbf{v}$. Thus the choice of origin defines a bijection

$$\begin{aligned} \mathsf{V} &\xrightarrow{A_o} \mathsf{E} \\ \mathbf{v} &\longmapsto o + \mathbf{v}. \end{aligned}$$

For any $o_1, o_2 \in \mathsf{E}$,

$$A_{o_1}(\mathbf{v}) = A_{o_2}\big(\mathbf{v} + (o_1 - o_2)\big)$$

where $o_1 - o_2 \in \mathsf{V}$ is the unique vector translating o_2 to o_1. A transformation $\mathsf{E} \xrightarrow{T} \mathsf{E}$ normalizes the group V of translations if and only if it is *affine,* that is, there is a linear transformation (denoted $\mathsf{L}(T)$, and called its *linear part*) such that, for a choice o of origin,

$$T(p) = o + \mathsf{L}(T)(p - o) + \mathbf{u}$$

for some vector $\mathbf{u} \in \mathsf{V}$ (called the *translational part of* T.)

2.2. **Causal structure.** The inner product induces a *causal structure* on V: a vector $\mathbf{v} \neq \mathsf{0}$ is called

- *timelike* if $\mathbf{v} \cdot \mathbf{v} < 0$,
- *null* (or *lightlike*) if $\mathbf{v} \cdot \mathbf{v} = 0$, or
- *spacelike* if $\mathbf{v} \cdot \mathbf{v} > 0$.

We will call the corresponding subsets of V respectively V_-,V_0 and V_+. The set V_0 of null vectors is called the *light cone.*

Say that vectors $\mathbf{u}, \mathbf{v} \in \mathsf{V}$ are *Lorentzian-perpendicular* if $\mathbf{u} \cdot \mathbf{v} = 0$. Denote the linear subspace of vectors Lorentzian-perpendicular to $\mathbf{v}$ by $\mathbf{v}^\perp$. A line $p + \mathbb{R}\mathbf{v}$ or ray $p + \mathbb{R}^+\mathbf{v}$ is called

- a *particle* if $\mathbf{v}$ is timelike,
- a *photon* if $\mathbf{v}$ is null, and
- a *tachyon* if $\mathbf{v}$ is spacelike.

The set of timelike vectors admits two connected components. Each component defines a *time-orientation* on V, and the time-orientation structure on V is carried on to E. We select one of the components and

call it Future. Call a non-spacelike vector $\mathbf{v} \neq \mathsf{0}$ and its corresponding ray *future-pointing* if $\mathbf{v}$ lies in the closure of Future.

The time-orientation can be defined by a choice of a timelike vector $\mathbf{t}$ as follows. Consider the linear functional $\mathsf{V} \longrightarrow \mathbb{R}$ defined by:

$$\mathbf{v} \longmapsto \mathbf{v} \cdot \mathbf{t}.$$

Then the future and past components can be distinguished by the sign of this functional on the set of timelike vectors.

2.2.1. *Null frames.* The restriction of the inner product to the orthogonal complement $\mathbf{s}^{\perp}$ of a spacelike vector $\mathbf{s}$ is indefinite, having signature $(1,1)$. The intersection of the light cone with $\mathbf{s}^{\perp}$ consists of two photons intersecting transversely at the origin. Choose a linearly independent pair of future-pointing null vectors $\mathbf{s}^{\pm} \in \mathbf{v}^{\perp}$ such that: $\{\mathbf{s}, \mathbf{s}^-, \mathbf{s}^+\}$ is a positively oriented basis for V (with respect to a fixed orientation on V). The null vectors $\mathbf{s}^-$ and $\mathbf{s}^+$ are defined only up to positive scaling. The standard identity (compare [13] , for example), for a unit spacelike vector $\mathbf{s}$

$$\begin{aligned} \mathbf{s} \times \mathbf{s}^- &= \mathbf{s}^- \\ \mathbf{s} \times \mathbf{s}^+ &= -\mathbf{s}^+, \end{aligned} \tag{3}$$

will be useful.

We call the positively oriented basis $\{\mathbf{s}, \mathbf{s}^-, \mathbf{s}^+\}$ a *null frame* associated to $\mathbf{s}$. Margulis [18, 19] takes the null vectors $\mathbf{s}^-, \mathbf{s}^+$ to have unit Euclidean length; in this paper we will not specify the vectors, except that they are future-pointing. In this basis the corresponding Gram matrix (the symmetric matrix of inner products) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}$$

and the nonzero entry (equal to $\frac{1}{2}\mathbf{s}^- \cdot \mathbf{s}^+$) is negative (since both $\mathbf{s}^-$ and $\mathbf{s}^+$ are future-pointing). Often we normalize $\mathbf{s}$ to be *unit-spacelike,* that is, $\mathbf{s} \cdot \mathbf{s} = 1$, and choose $\mathbf{s}^-$ and $\mathbf{s}^+$ so that $\mathbf{s}^- \cdot \mathbf{s}^+ = -1$.

The basis defines linear coordinates (a, b, c) on V:

$$\mathbf{v} := a\mathbf{s} + b\mathbf{s}^- + c\mathbf{s}^+$$

so the corresponding Lorentz metric on E is:

$$da^2 - db\ dc. \tag{4}$$

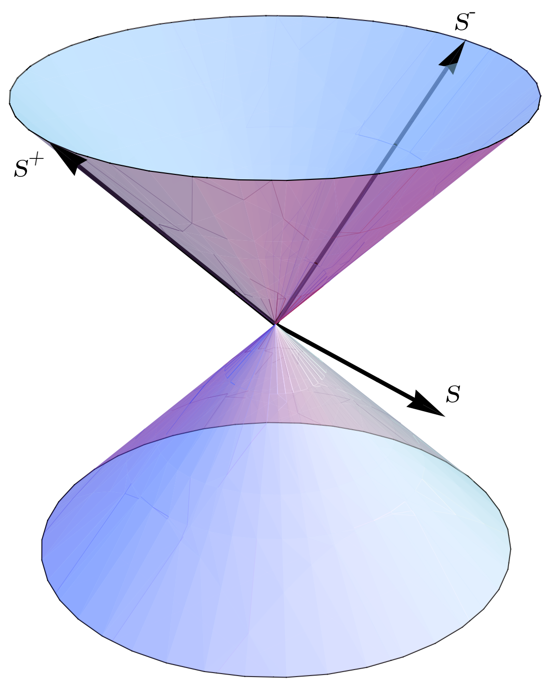

FIGURE 1. A null frame.

2.3. **Transformations of E.** The orientation on the vector space V defines an orientation on the manifold E. A linear automorphism of V preserves orientation if and only if it has positive determinant. An affine automorphism of E preserves orientation if and only if its linear part lies in the subgroup $\mathsf{GL}^+(3,\mathbb{R})$ of $\mathsf{GL}(3,\mathbb{R})$ consisting of matrices of positive determinant. The group of orientation-preserving affine automorphisms of E then decomposes as a semidirect product:

$$\mathsf{Aff}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \mathsf{GL}^+(3,\mathbb{R}).$$

Denote the group of orthogonal automorphisms (linear isometries) of V by $\mathsf{O}(2,1)$. Let

$$\mathsf{SO}(2,1) := \mathsf{O}(2,1) \cap \mathsf{GL}^+(3,\mathbb{R})$$

denote, as usual, the subgroup of orientation-preserving linear isometries. Orientation-preserving isometries of E constitute the subgroup:

$$\mathsf{Isom}^+(\mathsf{E}) \;:=\; \mathsf{V} \rtimes \mathsf{SO}(2,1)$$

and the subgroup of orientation-preserving conformal automorphisms is:

$$\mathsf{Conf}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \big(\mathsf{SO}(2,1) \times \mathbb{R}^+\big)$$

2.3.1. *Components of the isometry group.* The group $\mathsf{O}(2,1)$ has four connected components. The identity component $\mathsf{SO}^0(2,1)$ consists of orientation-preserving linear isometries preserving time-orientation. It is isomorphic to the group $\mathsf{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane H^2. (The relationship with hyperbolic geometry will be explored in §2.5.) The group $\mathsf{O}(2,1)$ is a semidirect product

$$\mathsf{O}(2,1) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathsf{SO}^0(2,1)$$

where $\pi_0\big(\mathsf{O}(2,1)\big) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by reflection in a point (the antipodal map $\mathbb{A}$, which reverses orientation) and reflection in a tachyon (which preserves orientation, but reverses time-orientation).

2.3.2. *Transvections, boosts, homotheties and reflections.* In the null frame coordinates of §2.2.1 , the one-parameter group of linear isometries

$$\xi_t \;:=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \tag{5}$$

(for $t \in \mathbb{R}$) fixes $\mathbf{s}$ and acts on the (indefinite) plane $\mathbf{s}^\perp$. These transformations, called *boosts,* constitute the identity component $\mathsf{SO}^0(1,1)$ of the isometry group of $\mathbf{v}^\perp$.

The one-parameter group $\mathbb{R}^+$ of *positive homotheties*

$$\eta_s \;:=\; \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^s \end{bmatrix}$$

(where $s \in \mathbb{R}$) acts conformally on Minkowski space, preserving orientation. The involution

$$\rho \;:=\; \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{6}$$

preserves each preserves orientation, reverses time-orientation, reverses $\mathbf{s}$, and interchanges the two null lines $\mathbb{R}\mathbf{s}^-$ and $\mathbb{R}\mathbf{s}^+$.

2.4. **Octants, quadrants, solid quadrants.** The following terminology will be used in the sequel. A *quadrant* in a vector space V is the set of nonnegative linear combinations of two linearly independent vectors, and a *quadrant* in an affine space E as translate of a point in E by a quadrant in the vector space underlying E. Similarly an *octant* in a vector space or affine space is obtained from nonnegative linear combinations of *three* linearly independent vectors. A *solid quadrant* is the set of linear combinations $a\mathbf{a} + b\mathbf{b} + c\mathbf{c}$ where $a, b, c \in \mathbb{R}$, $a, b \geq 0$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent.

2.5. **Hyperbolic geometry.** The Klein-Beltrami projective model of hyperbolic geometry identifies the hyperbolic plane H^2 with the subset $\mathsf{P}(\mathsf{V}_-)$ of the real projective plane $\mathsf{P}(\mathsf{V})$ corresponding to particles (timelike lines). Fixing an origin $o \in \mathsf{E}$ identifies the affine (Minkowski) space E with the Lorentzian vector space V. Thus the hyperbolic plane identifies with particles passing through o, or equivalently translational equivalence classes (parallelism classes) of particles in E.

2.5.1. *Orientations in* H^2. An orientation of H^2 corresponds to a fixed orientation on V and a *time-orientation* in V, that is, a connected component of V_-, as follows:

Consider the subset

$$\mathsf{H}^2_{\mathsf{fut}} := \big\{ \mathbf{v} \in \mathsf{Future} \mid \mathbf{v} \cdot \mathbf{v} = -1 \big\}$$

of the selected connected component Future of V_-. It is a cross-section for the $\mathbb{R}^+$-action V_- by homotheties: the restriction of the quotient mapping

$$\mathsf{V} \setminus \{0\} \longrightarrow \mathsf{P}(\mathsf{V})$$

to $\mathsf{H}^2_{\mathsf{fut}}$ identifies $\mathsf{H}^2_{\mathsf{fut}} \xrightarrow{\cong} \mathsf{H}^2$.

The radial vector field on V is transverse to the hypersurface $\mathsf{H}^2_{\mathsf{fut}}$, and so that vector field, together with the ambient orientation on V, defines an orientation on H^2. Using the *past-pointing timelike vectors* for a model for H^2 along with the fixed orientation of V, we would obtain the opposite orientation. This follows since the antipodal map on V relates future and past, and the antipodal map reverses orientation (in dimension 3). Fixing a time-orientation and reversing orientation in V reverses the induced orientation on H^2.

2.5.2. *Halfplanes in* H^2. Just as points in H^2 correspond to particles in E, geodesics in H^2 admit a completely analogous description as translational equivalence classes of tachyons in E. Given a spacelike vector $\mathbf{v} \in \mathsf{V}$, the projectivization $\mathsf{P}(\mathbf{v}^\perp)$ meets $\mathsf{P}(\mathsf{V}_-) \approx \mathsf{H}^2$ in a geodesic. A geodesic in H^2 separates H^2 into two *halfplanes.*

With a time-orientation, spacelike vectors in V conveniently parametrize halfplanes in H^2. Using the identification of H^2 with $\mathsf{H}^2_{\mathsf{fut}}$ above, a spacelike vector $\mathbf{s}$ determines a *halfplane* in H^2:

$$\mathfrak{h}(\mathbf{s}) := \left\{\mathbf{v} \in \mathsf{H}^2_{\mathsf{fut}} \mid \mathbf{v}\cdot\mathbf{s} \geq 0\right\}$$

bounded by the geodesic $\mathsf{P}(\mathbf{s}^{\perp})$. The complement of the geodesic $\mathsf{P}(\mathbf{s}^{\perp})$ in H^2 consists of the interiors of the two halfplanes $\mathfrak{h}(\mathbf{s})$ and $\mathfrak{h}(-\mathbf{s})$.

The time-orientation on V (together with the orientation on V) induces an orientation on H^2. Thus an *oriented geodesic* $l \subset \mathsf{H}^2$ determines a halfplane whose boundary is l.

Transitivity of the action of $\mathsf{Isom}^+(\mathsf{H}^2)$ on oriented geodesics implies:

Lemma 2.1. *The group of orientation-preserving isometries of* H^2 *acts transitively on the set of halfplanes in* H^2*. The isotropy group of a halfplane* $\mathfrak{h}(\mathbf{s})$ *is the one-parameter group of transvections along the geodesic* $\partial\mathfrak{h}(\mathbf{s})$ *corresponding to the one-parameter group* $\{\xi_t, t \in \mathbb{R}\}$*.*

2.6. **Disjointness of halfplanes.** We use a disjointness criterion for two halfplanes in H^2 in terms of the following definition:

Definition 2.2. Two spacelike vectors $\mathbf{s}_1, \mathbf{s}_2 \in \mathsf{V}$ are *consistently oriented* if $\mathbf{s}_1 \cdot \mathbf{s}_2 < 0$, $\mathbf{s}_1 \cdot \mathbf{s}_2^{\pm} \leq 0$, and $\mathbf{s}_1^{\pm} \cdot \mathbf{s}_2 \leq 0$.

Given the orientation defined above on H^2, two consistently oriented unit-spacelike vectors have a useful characterization in terms of halfplanes.

Lemma 2.3. *Let* $\mathbf{s}_1, \mathbf{s}_2 \in \mathsf{V}$ *be spacelike vectors. The vectors* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *are consistently oriented if and only if the corresponding halfplanes* $\mathfrak{h}(\mathbf{s}_1)$ *and* $\mathfrak{h}(\mathbf{s}_2)$ *are disjoint.*

Proof. Given a spacelike vector $\mathbf{s}_i$, the solid quadrant defined by combinations $a\mathbf{s}_i + b\mathbf{s}_i^- + c\mathbf{s}_i^+$ where $a, b, c \in \mathbb{R}$ and $b, c > 0$ contains all of the future-pointing timelike vectors. Moreover, the octant

$$A(\mathbf{s}) = \{a\mathbf{s}_i + b\mathbf{s}_i^- + c\mathbf{s}_i^+ \mid a, b, c > 0\}$$

defines the halfplane $\mathfrak{h}(\mathbf{s}_i)$. In particular,

$$\mathfrak{h}(\mathbf{s}_i) = A(\mathbf{s}_i) \cap \mathsf{H}^2.$$

Suppose that $\mathbf{s}_1$ and $\mathbf{s}_2$ are consistently oriented spacelike vectors. By definition, any vector in $A(\mathbf{s}_1)$ is a positive linear combination of vectors whose inner product with $\mathbf{s}_2$ is negative, so its inner product with $\mathbf{s}_2$ is negative. Thus $A(\mathbf{s}_1) \cap A(\mathbf{s}_2) = \emptyset$ and $\mathfrak{h}(\mathbf{s}_1) \cap \mathfrak{h}(\mathbf{s}_2) = \emptyset$.

Now suppose that $\mathfrak{h}(\mathbf{s}_1) \cap \mathfrak{h}(\mathbf{s}_2) = \emptyset$. Then $A(\mathbf{s}_1) \cap A(\mathbf{s}_2) = \emptyset$, and $\mathbf{s}_i$ and $\mathbf{s}_i^{\pm}$ all have a negative inner product with $\mathbf{s}_j$, as desired. ☐

3. Crooked halfspaces

In this section we define crooked halfspaces and their basic structure. Following our earlier papers, we consider an *open* crooked halfspace $\mathcal{H}$, denoting its closure by $\overline{\mathcal{H}}$ and its complement by $\mathcal{H}^c$.

A crooked halfspace $\mathcal{H}$ is bounded by a crooked plane $\partial\mathcal{H}$. A crooked plane is a 2-dimensional polyhedron with 4 faces, which is homeomorphic to $\mathbb{R}^2$. It is non-differentiable along two lines (called *hinges*) meeting in a point (called the *vertex*). The hinges are null lines bounding null halfplanes (called *wings.*) The wings are connected by the union of two quadrants in the plane containing the hinges. Call the plane spanned by the two hinges the *stem plane* and denote it $\mathcal{S}$. The hinges are the only null lines contained in $\mathcal{S}$. The union of the hinges and all timelike lines in $\mathcal{S}$ forms the *stem.* The stem plane may be equivalently defined as the unique plane containing the stem.

3.1. **The crooked halfspace.** We explicitly compute a crooked halfspace in the coordinates (a, b, c) defined in §2.2.1. Recall that in those coordinates the Lorentzian metric tensor equals $da^2 - db\,dc$. The reader should see from the definition that all crooked halfspaces are $\mathsf{Isom}^+(\mathsf{E})$-equivalent.

3.1.1. *The director and the vertex.* Let $\mathbf{s} \in \mathsf{V}$ be a (unit-) spacelike vector and $p \in \mathsf{E}$. Then the *(open) crooked halfspace directed by* $\mathbf{s}$ *and vertexed at* p is the union:

$$\begin{aligned}\mathcal{H}(p,\mathbf{s}) := &\{q \in \mathsf{E} \,|\, (q-p)\cdot\mathbf{s}^+ < 0 \text{ and } (q-p)\cdot\mathbf{s} > 0\} \cup \\ &\{q \in \mathsf{E} \,|\, (q-p)\cdot\mathbf{s}^+ < 0, (q-p)\cdot\mathbf{s}^- > 0, \text{ and } (q-p)\cdot\mathbf{s} = 0\} \cup \\ &\{q \in \mathsf{E} \,|\, (q-p)\cdot\mathbf{s}^- > 0 \text{ and } (q-p)\cdot\mathbf{s} < 0\}\end{aligned} \tag{7}$$

Its closure is the *closed crooked halfspace* with director $\mathbf{s}$ and vertex p is the union:

$$\begin{aligned}\overline{\mathcal{H}}(p,\mathbf{s}) := &\{q \in \mathsf{E} \,|\, (q-p)\cdot\mathbf{s}^+ \le 0 \text{ and } (q-p)\cdot\mathbf{s} \ge 0\} \\ &\cup \{q \in \mathsf{E} \,|\, (q-p)\cdot\mathbf{s}^- \ge 0 \text{ and } (q-p)\cdot\mathbf{s} \le 0\}\end{aligned} \tag{8}$$

Write

$$p =: \mathsf{Vertex}\big(\overline{\mathcal{H}}(p,\mathbf{s})\big) \;=\; \mathsf{Vertex}\big(\mathcal{H}(p,\mathbf{s})\big).$$

The director and vertex of the halfspace are the unit-spacelike vector and the point

$$\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

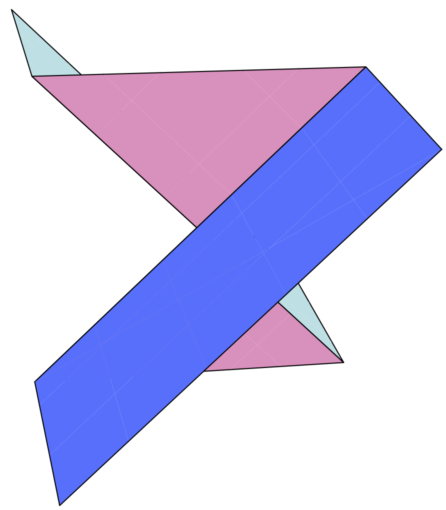

FIGURE 2. A crooked plane. Wings are halfplanes tangent to the null cone, and the stem are the two infinite timelike triangles.

respectively. Null vectors corresponding to $\mathbf{s}$ are:

$$\mathbf{s}^- = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{s}^+ = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

so that in above coordinates $\mathcal{H}$ is defined by the inequalities:

$$\begin{aligned} b > 0 & \quad \text{if } a > 0 \\ b > 0 > c & \quad \text{if } a = 0 \\ 0 > c & \quad \text{if } a < 0. \end{aligned}$$

The corresponding closed crooked halfspace $\overline{\mathcal{H}}$ is defined by:

$$\begin{aligned} b \geq 0 \quad &\text{if } a > 0 \\ b \geq 0 \text{ or } 0 \geq c \quad &\text{if } a = 0 \\ 0 \geq c \quad &\text{if } a < 0. \end{aligned}$$

3.1.2. *Octant notation.* The shape of a crooked halfspace suggests the following notation:

$$\overline{\mathcal{H}} = \{a, b \geq 0\} \cup \{a, c \leq 0\}.$$

The three coordinate planes for (a, b, c) divides E (identified with V) into eight open octants, depending on the signs of these three coordinates. Denote a subset of E by an ordered triple of symbols such as $+, -, 0, \pm$ to describe whether the corresponding coordinate is respectively positive, negative, zero, or arbitrary. For example the positive octant is $(+, +, +)$ and the negative octant is $(-, -, -)$. In this notation, the open crooked halfspace $\mathcal{H}(\mathbf{s}, p)$ is the union

$$(+, +, \pm) \cup (0, +, -) \cup (-, \pm, -)$$

3.1.3. *The hinges and the stem plane.* The *hinges* of $\mathcal{H}$ are the lines through the vertex parallel to the null vectors $\mathbf{s}^-, \mathbf{s}^+$:

$$\begin{aligned} h_-(p, \mathbf{s}) &:= p + \mathbb{R}\mathbf{s}^-, \\ h_+(p, \mathbf{s}) &:= p + \mathbb{R}\mathbf{s}^+. \end{aligned}$$

so the *stem plane* (the affine plane spanned by the hinges) equals:

$$\mathcal{S}(p, \mathbf{s}) := p + \mathbf{s}^\perp.$$

In octant notation, $h_- = (0, \pm, 0)$, $h_+ = (0, 0, \pm)$ and the stem plane is the coordinate plane $(0, \pm, \pm)$ defined by $a = 0$.

3.1.4. *The stem.* The stem consists of timelike directions inside the light cone in the stem plane. That is,

$$\mathsf{Stem}(p, \mathbf{s}) := \{p + \mathbf{v} \mid \mathbf{v} \in \mathbf{s}^\perp,\ \mathbf{v} \cdot \mathbf{v} \leq 0\}$$

In quadrant notation the stem is $(0, +, +) \cup (0, -, -)$ and is defined by:

$$a = 0,\ bc > 0.$$

The stem decomposes into two components: a *future* stem $(0, +, +)$ and a *past* stem $(0, -, -)$. Of course the boundary $\partial\mathsf{Stem}$ is the union of the hinges $h_- \cup h_+$.

3.1.5. *Particles in the stem.* Particles in the stem also determine involutions which interchange the pair of halfspaces complementary to $\mathcal{C}$. Particles are lines spanned by the future-pointing timelike vectors

$$\mathbf{t}_t := \begin{bmatrix} 0 \\ e^t \\ e^{-t} \end{bmatrix},$$

for any $t \in \mathbb{R}$, with the corresponding particles defined by $a = 0$, $b = e^{2t}c$. The corresponding reflection is:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xmapsto{R_t} \begin{bmatrix} -a \\ e^{2t}c \\ e^{-2t}b \end{bmatrix}$$

See [3] for a detailed study of involutions of E.

3.2. **The wings.** The wings are defined by a construction (denoted $\mathcal{W}$) involving the orientation of V. We associate to every null vector $\mathbf{n}$ a *null halfplane* $\mathcal{W}(\mathbf{n}) \subset \mathsf{V}$ and to every null line $p + \mathbb{R}\mathbf{n}$ the affine null halfplane $p + \mathcal{W}(\mathbf{n})$. Define the *wings* of the halfspace $\mathcal{H}(\mathbf{s}, p)$ as $p + \mathcal{W}(\mathbf{s}^-)$ and $p + \mathcal{W}(\mathbf{s}^+)$ respectively.

3.2.1. *Null halfplanes.* Let $\mathbf{n}$ be a future-pointing null vector. Its orthogonal plane $\mathbf{n}^\perp$ is tangent to the light cone. Then the line $\mathbb{R}\mathbf{n}$ lies in the plane $\mathbf{n}^\perp$. The complement $\mathbf{n}^\perp \setminus \mathbb{R}\mathbf{n}$ has two components, called *null halfplanes*. Consider a spacelike vector $\mathbf{v} \in \mathbf{n}^\perp$. Then $\mathbf{n}$ is either a multiple of $\mathbf{v}^+$ or $\mathbf{v}^-$. Two spacelike vectors $\mathbf{v},\ \mathbf{w} \in \mathbf{n}^\perp$ are in the same halfplane if and only if

$$\mathbf{v}^+ = \mathbf{w}^+ = \mathbf{n} \ \text{ or } \ \mathbf{v}^- = \mathbf{w}^- = \mathbf{n},$$

up to scaling by a positive real. A spacelike vector $\mathbf{s} \in \mathsf{V}$ thus unambiguously defines the following (positively extended) wing:

$$\mathcal{W}(\mathbf{s}) := \{\mathbf{w} \in \mathsf{V} \mid \mathbf{w} \cdot \mathbf{s} \geq 0 \text{ and } \mathbf{w} \cdot \mathbf{s}^+ = 0\} \tag{9}$$

Each hinge bounds a wing. The wings bounded by the hinges $h_- = (0, 0, \pm)$ and $h_+ = (0, \pm, 0)$ are defined, respectively, by:

$$\mathcal{W}_- := (+, 0, \pm) = \{a \geq 0, b = 0\},$$
$$\mathcal{W}_+ := (-, \pm, 0) = \{a \leq 0, c = 0\}.$$

3.2.2. *The spine.* A crooked plane $\mathcal{C}$ contains a unique tachyon σ called its *spine.* It lies in the union of the two wings, and is orthogonal to each hinge. The spine is defined by $b = c = 0$ or $(\pm, 0, 0)$ in quadrant notation.

Reflection R in the spine interchanges the halfspaces complementary to $\mathcal{C}$. In the usual coordinates it is:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \overset{R}{\longmapsto} \begin{bmatrix} a \\ -b \\ -c \end{bmatrix}. \tag{10}$$

Furthermore each halfspace complementary to $\mathcal{C}$ is a fundamental domain for $\langle R \rangle$.

3.2.3. *The role of orientation.* The orientation of E is crucially used to define wings. Since the group of all automorphisms of E is a double extension of the group of orientation-preserving automorphisms by the antipodal map $\mathbb{A}$, one obtains a parallel but opposite theory by composing with $\mathbb{A}$. (Alternatively, one could work with negatively oriented bases to define null frames etc.) *Negatively extended* crooked halfspaces and crooked planes are defined as in (7) except that all the inequalities involving $(q - p) \cdot \mathbf{v}$ are reversed. In this paper we fix the orientation of E and thus only consider positively extended halfspaces. For more details, see [13].

3.3. **The bounding crooked plane.** If $\mathcal{H}(p, \mathbf{s})$ is a halfspace, then its boundary $\partial\mathcal{H}(p, \mathbf{s})$ is a crooked plane, denoted $\mathcal{C}(p, \mathbf{s})$. A crooked plane is the union of its stem and two wings along the hinges which meet at the vertex. Observe that the complement of $\mathcal{H}(p, \mathbf{v})$ is the closed crooked halfspace $\overline{\mathcal{H}}(p, -\mathbf{v})$ and

$$\partial\mathcal{H}(p, \mathbf{v}) = \mathcal{C}(p, \mathbf{v}) = \partial\mathcal{H}(p, -\mathbf{v}).$$

3.4. **Transitivity.** For calculations it suffices to consider only one example of a crooked halfspace thanks to:

Lemma 3.1. *The group* $\mathsf{Isom}^+(\mathsf{E})$ *acts transitively on the set of (positively oriented) crooked halfspaces in* E.

Proof. The group V acts transitively on the set of points $p \in \mathsf{E}$ and by Lemma 2.1 $\mathsf{SO}(2, 1)$ acts transitively on the set of unit spacelike vectors $\mathbf{s}$. Thus $\mathsf{Isom}^+(\mathsf{E})$ acts transitively on the set of pairs $(p, \mathbf{s})$ where $p \in \mathsf{E}$ is a point and $\mathbf{s}$ is a unit-spacelike vector. Since such pairs determine crooked half spaces, $\mathsf{Isom}^+(\mathsf{E})$ acts transitively on crooked halfspaces. $\square$

In a similar way, the full group of (possibly orientation-reversing) isometries of E acts transitively on the set of (possibly negatively extended) crooked halfspaces.

3.5. **The stem quadrant.** A particularly important part of the structure of a crooked halfspace $\mathcal{H}$ is its *stem quadrant* $\mathsf{Quad}(\mathcal{H})$, defined as the closure of the intersection of $\mathcal{H}$ with its stem plane $\mathcal{S}(\mathcal{H})$ and denoted:

$$\mathsf{Quad}(\mathcal{H}) := \overline{\big(\mathcal{H} \cap \mathcal{S}(\mathcal{H})\big)} \subset \mathsf{E}. \tag{11}$$

Closely related is the *translational semigroup* $\mathsf{V}(\mathcal{H})$, defined as the set of translations preserving $\mathcal{H}$:

$$\mathsf{V}(\mathcal{H}) := \{\mathbf{v} \in \mathsf{V} \mid \mathcal{H} + \mathbf{v} \subset \mathcal{H}\} \subset \mathsf{V}. \tag{12}$$

Since the vertex o of a halfspace is uniquely determined by the halfspace, we choose o as an origin, and use the identification $\mathsf{V} \xrightarrow{A_o} \mathsf{E}$ to identify $\mathsf{Quad}(\mathcal{H})$ with a subset of E.

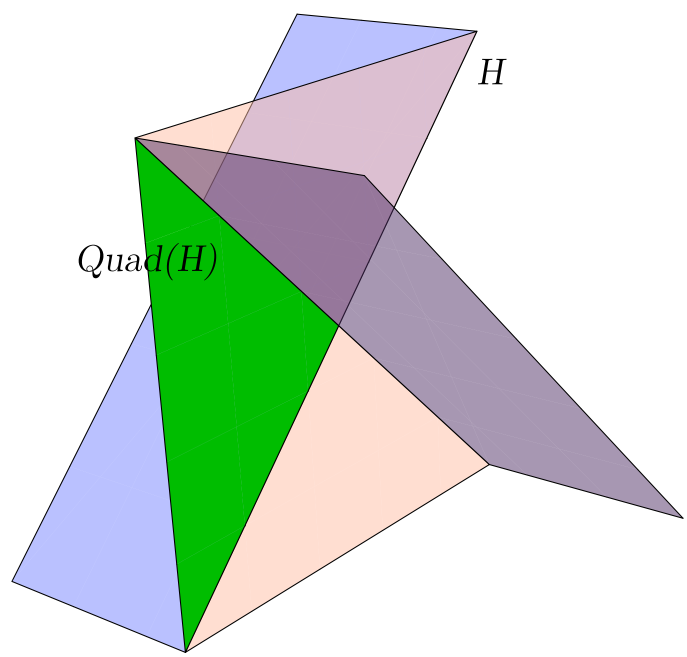

FIGURE 3. The stem quadrant of a crooked halfspace.

Theorem 3.2. *Let $\mathcal{H}$ be a crooked halfspace with vertex*

$$p := \mathsf{Vertex}(\mathcal{H}) \in \mathsf{E},$$

stem quadrant $\mathsf{Quad}(\mathcal{H}) \subset \mathsf{E}$*, and translational semigroup* $\mathsf{V}(\mathcal{H}) \subset \mathsf{V}$*. Then:*

$$\mathsf{Quad}(\mathcal{H}) \;=\; p + \mathsf{V}(\mathcal{H}).$$

The calculations in the proof will show that the $\mathsf{V}(\mathcal{H})$ has a particularly simple form:

Corollary 3.3. *Let* $\mathbf{s} \in \mathsf{V}$ *be a unit-spacelike vector and* $p \in \mathsf{E}$*. Then* $\mathsf{V}\big(\mathcal{H}(\mathbf{s}, p)\big)$ *consists of nonnegative linear combinations of* $\mathbf{s}^-$ *and* $-\mathbf{s}^+$*.*

Proof. Write the stem quadrant in the usual coordinates:

$$\mathsf{Quad}(\mathcal{H}) = \overline{(0, +, -)} = \{c \le 0 = a \le b\}.$$

A vector $\mathbf{v} = (\alpha, \beta, \gamma) \in \mathsf{V}$ satisfies $p + \mathbf{v} \in \mathsf{Quad}(\mathcal{H})$ if and only if:

$$\gamma \le 0 = \alpha \le \beta. \tag{13}$$

We first show that if $p + \mathbf{v} \in \mathsf{Quad}(\mathcal{H})$, then $\mathbf{v} \in \mathsf{V}(\mathcal{H})$. Suppose the coordinates α, β, γ of $\mathbf{v}$ satisfy (13) and let $p = (a, b, c) \in \mathcal{H}$.

- If $a > 0$, then $\alpha + a = a > 0$ and $\beta + b > 0$.
- If $a = 0$, then $\alpha + a = 0$ and $\beta + b > 0$, as well as $\gamma + c < 0$.
- If $a < 0$, then $\alpha + a = a < 0$ and $\gamma + c < 0$.

Thus $p + \mathbf{v} \in \mathcal{H}$ as desired.

Conversely, suppose that $\mathbf{v} \in \mathsf{V}(\mathcal{H})$. Suppose that $\alpha > 0$. Choose $a < -\alpha$ and $b < -|\beta|$. Then $p = (a, b, c) \in \mathcal{H}$ but $\mathbf{v} + p \notin \mathcal{H}$, a contradiction. If $\alpha < 0$, then taking $a > -\alpha$ and $c > |\gamma|$ leads to a contradiction. Thus $\alpha = 0$.

We next prove that $\beta \le 0$. Otherwise $\beta > 0$ and taking $p = (0, b, c)$ where $c < -\gamma$ and $b > 0$ yields a contradiction. Similarly $\beta \ge 0$. Thus (13) holds, proving $p + \mathbf{v} \in \mathsf{Quad}(\mathcal{H})$ as desired. ∎

Proposition 3.4. *Let* $\mathbf{s} \in \mathsf{V}$ *be a unit-spacelike vector and* $p \in \mathsf{E}$*. Then the complementary open halfspace* $\overline{\mathcal{H}(p, \mathbf{s})}^c$ *equals* $\mathcal{H}(p, -\mathbf{s})$ *and*

$$\mathsf{Quad}\big(\overline{\mathcal{H}(p, \mathbf{s})}^c\big) = -\mathsf{Quad}\big(\mathcal{H}(p, \mathbf{s})\big).$$

3.6. **Linearization of crooked halfspaces.** Recall that in §2.5.2 we associated every spacelike vector in V to a halfplane in H^2. Given $\mathbf{s} \in \mathsf{V}$ spacelike and $p \in \mathsf{E}$, we define the *linearization* of $\mathcal{H}(p, \mathbf{s})$ to be:

$$\mathsf{L}(\mathcal{H}(p, \mathbf{s})) = \mathfrak{h}(\mathbf{s})$$

We first show that linearization commutes with complement:

Corollary 3.5. *Suppose $\mathcal{H} \subset \mathsf{E}$ is a crooked halfspace with complementary halfspace $\mathcal{H}^c$. Then the linearization $\mathsf{L}(\mathcal{H}^c)$ is the halfplane in H^2 complementary to $\mathsf{L}(\mathcal{H})$.*

Proof. The spine reflection R defined in (10), §3.2.2 interchanges $\mathcal{H}$ and $\mathcal{H}^c$. Furthermore its linearization $\mathsf{L}(R)$ is a reflection in $\partial\mathsf{L}(\mathcal{H})$ which interchanges the particles in $\mathcal{H}$ and $\mathcal{H}^c$. Therefore $\mathsf{L}(\mathcal{H}^c)$ and $\mathsf{L}(\mathcal{H})$ are complementary halfplanes in H^2 as claimed. □

Next we deduce that linearization preserves the relation of inclusion of halfspaces.

Corollary 3.6. *Suppose that $\mathcal{H}_1, \mathcal{H}_2 \subset \mathsf{E}$ are crooked halfspaces, with linearizations $\mathsf{L}(\mathcal{H}_1), \mathsf{L}(\mathcal{H}_2) \subset \mathsf{H}^2$. Then the correspondence L is order-preserving:*

$$\mathcal{H}_1 \subset \mathcal{H}_2 \implies \mathsf{L}(\mathcal{H}_1) \subset \mathsf{L}(\mathcal{H}_2).$$

Proof. Let $\mathbf{t} \in \mathsf{L}(\mathcal{H}_1)$. Then there exists a particle ℓ parallel to $\mathbf{t}$ such that $\ell \subset \mathcal{H}_1$. Since $\mathcal{H}_1 \subset \mathcal{H}_2$, the particle ℓ lies in $\mathcal{H}_2$. Thus $\mathbf{t} \in \mathsf{L}(\mathcal{H}_2)$ as claimed. □

Clearly $\mathsf{L}(\mathcal{H}_1) \subset \mathsf{L}(\mathcal{H}_2)$ does *not* in general imply that $\mathcal{H}_1 \subset \mathcal{H}_2$.

Corollary 3.7. *Suppose $\mathcal{H}_1, \mathcal{H}_2$ are disjoint crooked halfspaces. Then their linearizations $\mathsf{L}(\mathcal{H}_1), \mathsf{L}(\mathcal{H}_2)$ are disjoint halfplanes in H^2.*

Proof. Combine Corollary 3.5 and Corollary 3.6. □

4. Symmetry

In this section we determine various automorphism groups and endomorphism semigroups of a crooked halfspace and the corresponding orbit structure.

We begin by decomposing a halfspace into pieces, which will be invariant under the affine transformations. From that we specialize to conformal automorphisms, and finally isometries.

4.1. **Decomposing a halfspace.** The open halfspace $\mathcal{H} = \mathcal{H}(p, \mathbf{s})$ naturally divides into three subsets, the stem quadrant, defined by $a = 0$ in null frame coordinates, and two *solid quadrants,* defined by $a < 0$ and $a > 0$. A *solid quadrant* in a 3-dimensional affine space is defined as the intersection of two ordinary (parallel, that is, "non-crooked") halfspaces. Equivalently a solid quadrant is a connected component of the complement of the union of two transverse planes in E.

4.2. **Affine automorphisms.** We first determine the group of affine automorphisms of $\mathcal{H}$.

First, every automorphism g of $\mathcal{H}$ must fix $p := \mathsf{Vertex}(\mathcal{H})$ and preserve the hinges h_-, h_+. The crooked plane $\partial\mathcal{H}$ is smooth except along $h_- \cup h_+$ so g leaves this set invariant. Furthermore this set is singular only at the vertex $p = h_- \cap h_+$,

Since $\mathcal{H}$ is vertexed at p, the affine automorphism g must be linear (where p is identified with the zero element of V, of course).

The involution

$$\rho\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -a \\ -c \\ -b \end{bmatrix}$$

defined in (6) preserves $\mathcal{H}$ (and also $\bar{\mathcal{H}}^c$), but interchanges h_- and h_+. The involution preserves the particle

$$a = b + c = 0.$$

Thus, either g or $g\rho$ will preserve h_+ and h_-. We henceforth assume that g preserves each hinge.

The complement of h_- in $\partial\mathcal{H}$ has two components, one of which is smooth and the other singular (along h_+). The smooth component is the wing $\mathcal{W}_-$, which must be preserved by g. Thus g preserves each wing.

Each wing lies in a unique (null) plane, and these two null planes intersect in the spine defined in (10) in §3.2.2, the tachyon through o parallel to $\mathbf{s}$. The spine is also preserved by g. Thus g is represented by a linear map preserving the coordinate lines for the null frame $(\mathbf{s}, \mathbf{s}^-, \mathbf{s}^+)$, and therefore represented by a diagonal matrix. We have proved:

Proposition 4.1. *The affine automorphism group of $\mathcal{H}$ equals the double extension of the group of positive diagonal matrices by the order two cyclic group $\langle\rho\rangle$. It is the image of the embedding*

$$\begin{aligned} \mathbb{R}^3 \rtimes (\mathbb{Z}/2) &\xrightarrow{\cong} \mathsf{Aff}^+(\mathsf{E}) \\ \big((s,t,u),\epsilon\big) &\longmapsto \rho^\epsilon e^s \begin{bmatrix} e^u & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}. \end{aligned}$$

where $\epsilon \equiv 0, 1 \big(\mathrm{mod}\ 2\big)$.

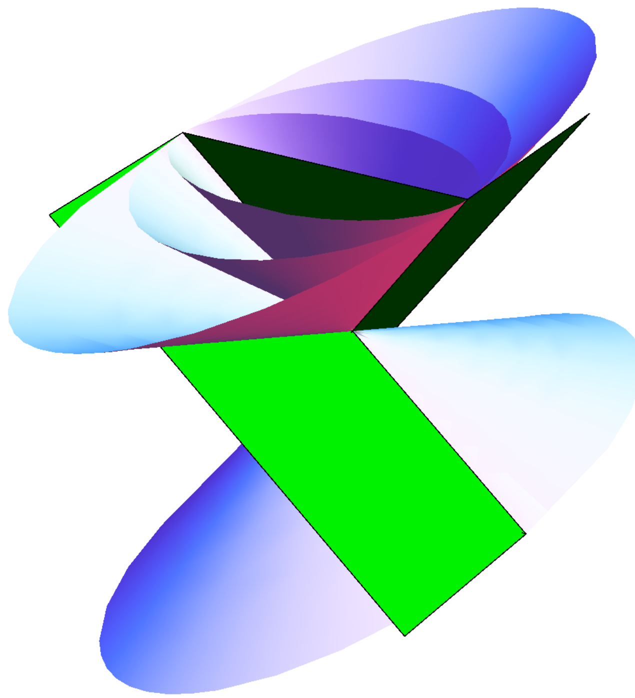

FIGURE 4. Affine automorphisms. This figure depicts a single crooked plane C, and the lightcones for three affinely equivalent Lorentz structures in which C is defined. For each of these Lorentz structures, a crooked halfspace complementary to C meets the future in a region defining a halfplane in H^2.

4.3. **Conformal automorphisms and isometries.** Lorentz isometries and homotheties generate the group of conformal automorphisms of E (*Lorentz similarity transformations*) By §2.3 a conformal transformation (respectively isometry) is an affine automorphism g whose linear part $\mathsf{L}(g)$ lies in $\mathsf{SO}(2,1)\times\mathbb{R}^+$ (respectively $\mathsf{SO}(2,1)$). By Proposition 4.1, the linear part $\mathsf{L}(g)$ is a diagonal matrix (in the null frame) so it suffices to check which diagonal matrices act conformally (respectively isometrically).

Proposition 4.2. *Let $\mathcal{H}$ be a crooked halfspace.*

- *The group* $\mathsf{Conf}^+(\mathcal{H})$ *of conformal automorphisms of* $\mathcal{H}$ *equals the double extension by the order two cyclic group* $\langle\rho\rangle$ *of the subgroup of positive diagonal matrices generated by positive homotheties and the one-parameter subgroup* $\{\eta_t \mid t \in \mathbb{R}\}$ *of boosts. It is the image of the embedding*
$$\mathbb{R}^2 \rtimes (\mathbb{Z}/2) \xrightarrow{\cong} \mathsf{Conf}^+(\mathcal{H})$$
$$\big((s,t),\epsilon\big) \longmapsto \rho^\epsilon e^s \eta_t$$
where $\epsilon \equiv 0, 1\big(\text{mod } 2\big)$.
- *The isometry group of* $\mathcal{H}$ *equals the double extension of the one-parameter subgroup* $\{\eta_t \mid t \in \mathbb{R}\}$ *of boosts, by the order two cyclic group* $\langle\rho\rangle$.

4.4. **Orbit structure.** In this section we describe the orbit space of $\mathcal{H}$ under the action of its conformal automorphism group $\mathsf{Conf}^+(\mathcal{H})$. The main goal is that the action is proper with orbit space homeomorphic to a half-closed interval. The function
$$\Phi(a,b,c) := bc/a^2$$
defines a homeomorphism of the orbit space with $\mathbb{R} \cup \{-\infty\}$. The action is not free, and the only fixed points are rays in the stem quadrant which are fixed under conjugates of the involution ρ.

4.4.1. *Action on the stem quadrant.*

Lemma 4.3. *The identity component* $\mathsf{Conf}^0(\mathcal{H}) \cong \mathbb{R}^2$ *acts transitively and freely on the stem quadrant* $\mathsf{Quad}(\mathcal{H})$.

Proof. Fix a basepoint q_0 in the stem quadrant:
$$q_0 := \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{14}$$
Then
$$\eta_s\xi_t(q_0) = \begin{bmatrix} 0 \\ e^{s+t} \\ -e^{s-t} \end{bmatrix}$$
An arbitrary point in the stem quadrant $\mathsf{Quad}(\mathcal{H})$ is:
$$p = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{15}$$

with $a = 0$ and $b \geq 0 \geq c$. Then

$$\begin{aligned} p &= \sqrt{-bc} \begin{bmatrix} 0 \\ \sqrt{-b/c} \\ -\sqrt{-c/b} \end{bmatrix} \\ &= \eta_s \xi_t(q_0) \end{aligned}$$

where

$$\begin{aligned} s &= \frac{\log(b) + \log(-c)}{2}, \\ t &= \frac{\log(b) - \log(-c)}{2} \end{aligned}$$

are uniquely determined. $\square$

However, the group of similarities $\mathsf{Conf}^+(\mathcal{H}) = \mathsf{Conf}^0(\mathcal{H}) \rtimes \langle \rho \rangle$ does *not* act freely on the stem quadrant as the involution ρ fixes the ray

$$\mathsf{Fix}(\rho) := \left\{ \begin{bmatrix} 0 \\ b \\ -b \end{bmatrix} \;\middle|\; b > 0 \right\}.$$

4.4.2. *Action on the solid quadrants.* The stem quadrant $\mathsf{Quad}(\mathcal{H})$ divides $\mathcal{H}$ into two solid quadrants, $(+,+,\pm)$ defined by $a > 0$, and $(-,\pm,-)$ defined by $a < 0$.

Lemma 4.4. *The identity component $\mathsf{Conf}^0(\mathcal{H})$ acts properly and freely on each solid quadrant in $\mathcal{H} \setminus \mathsf{Quad}(\mathcal{H})$, and ρ interchanges them. The function*

$$\begin{aligned} \mathcal{H} \setminus \mathsf{Quad}(\mathcal{H}) &\xrightarrow{\Phi} \mathbb{R} \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} &\longmapsto bc/a^2 \end{aligned}$$

defines a diffeomorphism

$$\big(\mathcal{H} \setminus \mathsf{Quad}(\mathcal{H})\big)/\mathsf{Conf}^+(\mathcal{H}) \xrightarrow{\approx} \mathbb{R}.$$

Proof. We only consider the solid quadrant $(+,+,\pm)$, since $(-,\pm,-)$ follows from this case by applying ρ.

We show that the set B of all

$$p_\gamma := \begin{bmatrix} 1 \\ 1 \\ \gamma \end{bmatrix}, \tag{16}$$

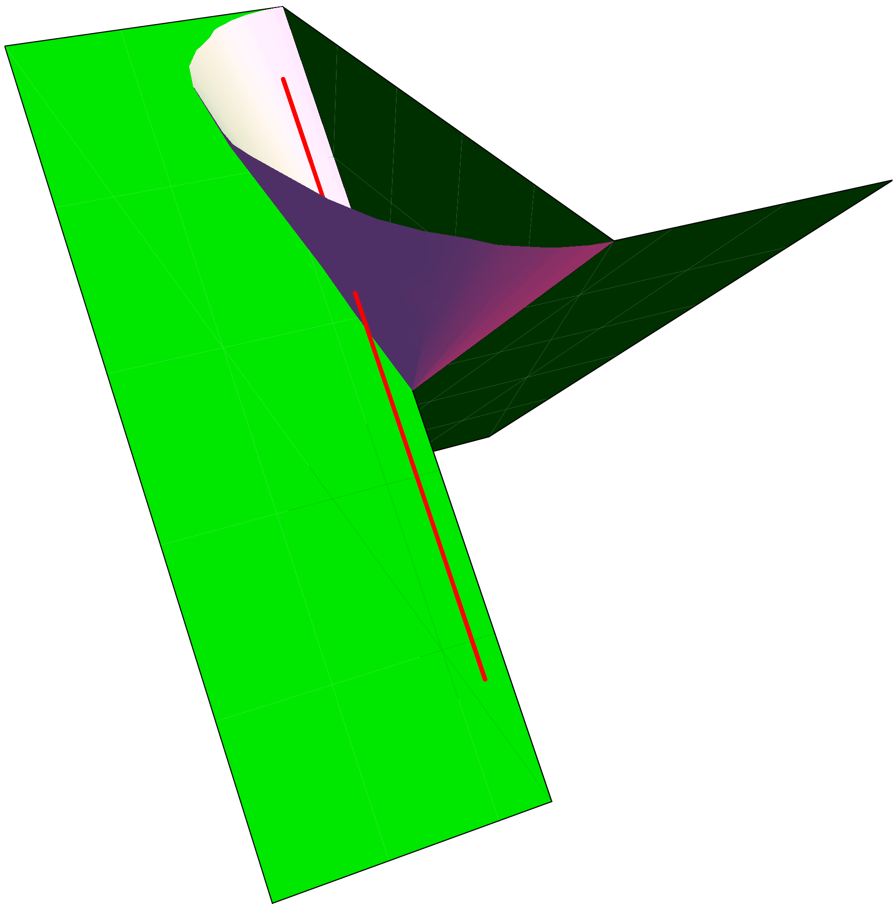

FIGURE 5. A slice of a basic orbit. The line is transverse to all of the orbits.

where $\gamma \in \mathbb{R}$, is a slice for the action on $(+,+,\pm)$. Namely, the map

$$\begin{aligned} \mathsf{Conf}^0(\mathcal{H}) \times B &\longrightarrow (+,+,\pm) \subset \mathcal{H} \\ \big((\eta_s,\xi_t),p_\beta\big) &\longmapsto \eta_s\xi_t(p_\beta) \end{aligned}$$

is a diffeomorphism. If p is an arbitrary point as in (15) above, and $a, c > 0$, then

$$\begin{aligned} s &:= \log(a) \\ t &:= \log(b/a) \\ \beta &:= \Phi(p) = bc/a^2 \end{aligned}$$

uniquely solves

$$\eta_s\xi_t(p_\beta) = p$$

and defines the smooth inverse map. Thus $\mathsf{Conf}^0(\mathcal{H})$ acts properly and freely on each solid quadrant. Since ρ interchanges these quadrants, $\mathsf{Conf}^+(\mathcal{H}) = \mathsf{Conf}^0(\mathcal{H}) \rtimes \langle\rho\rangle$ acts properly and freely on $\mathcal{H} \setminus \mathsf{Quad}(\mathcal{H})$ as claimed and Φ defines a quotient map. ☐

4.4.3. *Putting it all together.* Now combine Lemmas 4.3 and 4.4 to prove that $\mathsf{Conf}^+(\mathcal{H})$ acts properly on $\mathcal{H}$ with quotient map Φ onto the infinite half-closed interval $\{-\infty\} \cup \mathbb{R}$.

The main problem is that the slice B used in the proof of Lemma 4.4 does not extend to $\mathsf{Quad}(\mathcal{H})$, since $a \equiv 1$ on B and $\mathsf{Quad}(\mathcal{H})$ is defined by $a = 0$. To this end we replace the slice B, for parameter values $1 \geq a > 0$, by an equivalent slice which converges to the basepoint $q_0 \in \mathsf{Quad}(\mathcal{H})$ defined in (14). The point p_0 on B corresponding to parameter value $\beta = 0$, and the basepoint on $\mathsf{Quad}(\mathcal{H})$ equal

$$p_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad q_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

respectively. Thus we replace the segment of the slice B for $\beta \leq 0$, by points of the form

$$p'_a := \begin{bmatrix} a \\ 1 \\ a-1 \end{bmatrix}$$

for $1 \geq a > 0$. The corresponding γ-parameter is:

$$\gamma(a) := \Phi(p'_a) = \frac{a-1}{a^2}$$

with inverse function:

$$a(\gamma) := \frac{1 - \sqrt{1-4\gamma}}{2\gamma}$$

We obtain inverse diffeomorphisms

$$(0,1) \xrightarrow{\gamma} (-\infty, 0), \quad (-\infty, 0) \xrightarrow{a} (0,1)$$

which extend to homeomorphisms $[0,1] \approx [-\infty, 0]$.

4.4.4. *A global slice.* Thus we construct a slice σ for the $\mathsf{Conf}^0(\mathcal{H})$-action on $\mathcal{H}$ using the function $\gamma = \Phi(p)$ extended to

$$\mathcal{H} \xrightarrow{\Phi} \{-\infty\} \cup \mathbb{R}$$

by sending $\mathsf{Quad}(\mathcal{H})$ to $-\infty$. Furthermore we can extend σ uniquely to a ρ-equivariant slice for the action of $\mathsf{Conf}^0(\mathcal{H})$ on $\mathcal{H}$. We define the

continuous slice for the parameter $-\infty < a < \infty$; it is smooth except for parameter values $a = -1, 0, 1$ where it equals:

$$\sigma(-1) := \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \sigma(0) := q_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \sigma(1) := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

On the intervals $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, \infty)$, smoothly interpolate between these values:

$$\begin{aligned}
&\text{For } a \leq -1, && \sigma(a) := \begin{bmatrix} -1 \\ -\gamma(-a) \\ -1 \end{bmatrix} \\
&\text{For } -1 \leq a \leq 0, && \sigma(a) := \begin{bmatrix} a \\ a+1 \\ -1 \end{bmatrix} \\
&\text{For } 0 \leq a \leq 1, && \sigma(a) := \begin{bmatrix} a \\ 1 \\ a-1 \end{bmatrix} \\
&\text{For } 1 \leq a, && \sigma(a) := \begin{bmatrix} 1 \\ 1 \\ \gamma(a) \end{bmatrix}
\end{aligned}$$

5. Lines in a halfspace

In this section we classify the lines which lie entirely in a crooked halfspace. The natural context to initiate this question is affine; we develop a criterion in terms of the stem quadrant for a line to lie in a halfspace.

5.1. **Affine lines.** Given an affine line $\ell \subset \mathsf{E}$ and a point $o \in \mathsf{E}$, a unique line, denoted ℓ_o, is parallel to ℓ and contains o. Moreover if $\eta_t^{(o)}$ denotes the one-parameter group of homotheties fixing o and preserving the crooked halfspace, that is,

$$\begin{aligned} \mathsf{E} &\xrightarrow{\eta_t^{(o)}} \mathsf{E} \\ o + \mathbf{v} &\longmapsto o + e^t \mathbf{v}, \end{aligned}$$

then

$$\ell_o = \lim_{t \to -\infty} \eta_t^{(o)}(\ell). \tag{17}$$

Lemma 5.1. *If $\ell \subset \mathcal{H}$ and $o = \mathsf{Vertex}\mathcal{H}$, then $\ell_o \subset \overline{\mathcal{H}}$.*

Proof. The homotheties $\eta_t^{(o)} \in \mathsf{Aff}^+(\mathcal{H})$ so $\eta_t^{(o)}(\ell) \subset \mathcal{H}$. Apply (17) to the closed set $\overline{\mathcal{H}}$ to conclude that $\ell_o \subset \overline{\mathcal{H}}$. □

Now let $\mathcal{S} \subset \mathsf{E}$ be the stem plane of $\mathcal{H}$. Unless ℓ is parallel to $\mathcal{S}$, it meets $\mathcal{S}$ in a unique point p. Since $\ell \subset \mathcal{H}$ and

$$\overline{\mathcal{H} \cap \mathcal{S}} = \mathsf{Quad}(\mathcal{H}),$$

the stem quadrant $\mathsf{Quad}(\mathcal{H}) \ni p$. Then translation of ℓ_o by $p - o$ is a line through p which is parallel to ℓ, and thus equals ℓ. Therefore:

Lemma 5.2. *Every line contained in $\bar{\mathcal{H}}$ not parallel to $\mathcal{S}$ is the translate of a line in $\bar{\mathcal{H}}$ passing through* $\mathsf{Vertex}(\mathcal{H})$ *by a vector in* $\mathsf{V}(\mathcal{H})$.

5.1.1. *Lines through the vertex.* Now we determine when a line ℓ translated by a nonzero vector lies in $\overline{\mathcal{H}}$. Suppose that ℓ is spanned by the vector:

$$\mathbf{v} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \tag{18}$$

We shall use the *orthogonal projection* to the stem plane $\mathcal{S}$ defined by:

$$\mathsf{E} \longrightarrow \mathcal{S} \subset \mathsf{E}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}.$$

First suppose that ℓ doesn't lie in the stem plane $\mathcal{S}$, that is, $\alpha \neq 0$. By scaling, assume that $\alpha = 1$. Then ℓ consists of all vectors

$$a\mathbf{v} = \begin{bmatrix} a \\ a\beta \\ a\gamma \end{bmatrix},$$

where $a \in \mathbb{R}$. Since $\mathcal{H} = (+,+,\pm) \cup (0,+,-) \cup (-,\pm,-)$, the condition that $a\mathbf{v} \in \bar{\mathcal{H}}$ is equivalent to the two conditions:

- $a < 0$ implies $a\gamma \leq 0$;
- $a > 0$ implies $a\beta \geq 0$.

Thus $\ell \subset \overline{\mathcal{H}}$ if and only if $\beta, \gamma \geq 0$. Moreover $\ell \cap \overline{\mathcal{H}} = \{o\}$ if and only if $\beta, \gamma < 0$.

It remains to consider the case when $\ell \subset \mathcal{S}$, that is, $\alpha = 0$. Since

$$\mathsf{Stem}(\mathcal{H}) \;=\; \overline{(0,+,+) \cup (0,-,-)},$$

the condition that $\ell \subset \bar{\mathcal{H}}$ is equivalent to the conditions $\beta\gamma \geq 0$, that is, ℓ lies in a solid quadrant in $\bar{H}$ whose projection to $\mathcal{S}$ maps to $\mathsf{Stem}(\mathcal{H}) \subset \mathcal{S}$. We have proved:

Lemma 5.3. *Suppose $\ell \subset \bar{\mathcal{H}}$ is a line. Then orthogonal projection to $\mathcal{S}$ maps ℓ to $\overline{\mathsf{Stem}(\mathcal{H})} \subset \mathcal{S}$.*

5.2. **Lines contained in a halfspace and linearization.** Suppose that $\mathbf{v}$ defined in (18) is future-pointing timelike, and that $\mathbb{R}\mathbf{v} \subset \overline{\mathcal{H}}$. Then the discussion in §5.1.1 implies that necessarily $\beta, \gamma \geq 0$. Now $\alpha^2 < \beta\gamma$ implies that $\beta, \gamma > 0$, that is, that all coordinates α, β, γ have the same (nonzero) sign. This condition is equivalent to the future-pointing timelike vector having positive inner product with the unit-spacelike vector

$$\mathbf{s}_0 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and this condition defines a *halfplane* in H^2. We conclude:

Theorem 5.4. *Let $\mathcal{H}(\mathbf{s}, p) \subset \mathsf{E}$ be a crooked halfspace. Then the collection of all future-pointing unit-timelike vectors parallel to a particle contained in $\mathcal{H}(\mathbf{s}, p)$ is the halfplane $\mathfrak{h}(\mathbf{s}) \subset \mathsf{H}^2$.*

5.3. **Unions of particles.** We close this section with a converse statement.

Theorem 5.5. *Let $\mathcal{H} \subset \mathsf{E}$ be a crooked halfspace. Then every $p \in \mathcal{H}$ lies on a particle contained in $\mathcal{H}$.*

Proof. It suffices to prove the theorem for the halfspace $\mathcal{H} = (+,+,\pm) \cup (0,+,-) \cup (-,\pm -)$.

Start with any p in the open quadrant $(+,+,\pm)$ We will now describe a point q in the stem quadrant $(0,+,-)$ for which the vector $p-q$ is timelike. Write

$$p = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \ q = \begin{pmatrix} 0 \\ B \\ C \end{pmatrix}$$

so that $a, b > 0$ and $B > 0 > C$. First choose B so that $0 < B < b$, then choose

$$C < \min\left(0, c - \frac{a^2}{b-B}\right).$$

Let

$$\mathbf{v} = p - q = \begin{bmatrix} a \\ b - B \\ c - C \end{bmatrix}$$

so that

$$\mathbf{v} \cdot \mathbf{v} = a^2 - (b-B)(c-C) < a^2 - a^2 = 0,$$

$b - B > 0$ and $c - C > 0$. That is, the line $q + t\mathbf{v}$ is a particle. All of the points on the line where $t > 0$ lie inside the quadrant $(+,+,\pm)$ and all of the points where $t < 0$ lie inside the quadrant $(-,\pm,-)$.

A similar calculation applies to points in the $(-,\pm,-)$ quadrant.

It is left only to consider points $p \in \mathsf{Quad}(\mathcal{H}) = (0,+,-)$. Any timelike vector pointing inside of $\mathcal{H}$ will do, but choose the timelike vector

$$\mathbf{v} = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}.$$

Consider the line

$$p + t\mathbf{v} = \begin{pmatrix} t/2 \\ b+t \\ c+t \end{pmatrix}.$$

All points where $t > 0$ lie inside $(+,+,\pm)$, and all points where $t < 0$ lie inside $(-,\pm,-)$.

$\square$

6. Disjointness criteria

In this section we revisit the theory developed in [13] in terms of the notion of stem quadrants and crooked halfspaces. If $\mathcal{H}_1, \mathcal{H}_2$ are disjoint crooked halfspaces, then their linearizations $\mathfrak{h}_i = \mathsf{L}(\mathcal{H}_i)$ are disjoint halfplanes in H^2 (Corollary 3.7). Suppose that $\mathbf{s}_i$ is the spacelike vector corresponding to $\mathfrak{h}_i$ as in §2.5.2 and that they are consistently oriented.

Definition 6.1. Let $\mathbf{s}_1, \mathbf{s}_2$ be consistently oriented spacelike vectors. The interior of $\mathsf{V}(\mathbf{s}_1) - \mathsf{V}(\mathbf{s}_2)$ is called the cone of *allowable translations*, denoted $\mathsf{A}(\mathbf{s}_1, \mathbf{s}_2)$.

We show that two (open) crooked halfspaces with disjoint linearizations are disjoint if and only if the vector between their vertices lies in the closure of the cone of allowable translations.

Theorem 6.2. *Suppose that $\mathbf{s}_i$ are consistently oriented unit-spacelike vectors and $p_1, p_2 \in \mathsf{E}$. Then the closed crooked halfspaces $\overline{\mathcal{H}(p_1,\mathbf{s}_1)}$ and $\overline{\mathcal{H}(p_2,\mathbf{s}_2)}$ are disjoint if and only if*

$$p_1 - p_2 \in \mathsf{A}(\mathbf{s}_1, \mathbf{s}_2) \tag{19}$$

Similarly $\mathcal{H}(p_1,\mathbf{s}_1) \cap \mathcal{H}(p_2,\mathbf{s}_2) = \emptyset$ if and only if $p_1 - p_2$ lies in the closure of $\mathsf{A}(\mathbf{s}_1, \mathbf{s}_2)$.

Proof. We first show that (19) implies that $\bar{\mathcal{H}}(p_1,\mathbf{s}_1) \cap \bar{\mathcal{H}}(p_2,\mathbf{s}_2) = \emptyset$. Choose $\mathbf{v}_i \in \mathsf{V}(\mathbf{s}_i)$ for $i = 1,2$ respectively. Choose an arbitrary origin $p_0 \in \mathsf{E}$ and let $p_i := p_0 + \mathbf{v}_i$.

Lemma 2.3 implies that the crooked halfspaces $\mathcal{H}(p_0, \mathbf{s}_1)$ and $\mathcal{H}(p_0, \mathbf{s}_2)$ are disjoint. By Theorem 3.2,

$$\mathcal{H}(p_i, \mathbf{s}_i) := \mathcal{H}(p_0, \mathbf{s}_i) + \mathbf{v}_i \subset \mathcal{H}_i$$

Thus $\mathcal{H}(p_1, \mathbf{s}_1)$ and $\mathcal{H}(p_2, \mathbf{s}_2)$ are disjoint.

Conversely, suppose that $\mathcal{H}(p_1, \mathbf{s}_1) \cap \mathcal{H}(p_2, \mathbf{s}_2) = \emptyset$. We use the following results from [13], (Theorem 6.2.1 and Theorem 6.4.1), which are proved using a case-by-case analysis of intersections of wings and stems:

Proposition 6.3. *Let $\mathbf{s}_i \in \mathsf{V}$ be consistently oriented unit-spacelike vectors and $p_i \in \mathsf{E}$, for $i = 1, 2$. Then $\mathcal{C}(p_1, \mathbf{s}_1) \cap \mathcal{C}(p_2, \mathbf{s}_2) = \emptyset$ if and only if:*

- *for ultraparallel $\mathbf{s}_1$ and $\mathbf{s}_2$,*

$$(p_2 - p_1) \cdot (\mathbf{s}_1 \times \mathbf{s}_2) > |(p_2 - p_1) \cdot \mathbf{s}_1| + |(p_2 - p_1) \cdot \mathbf{s}_2| \tag{20}$$

- *for asymptotic $\mathbf{s}_1$ and $\mathbf{s}_2$ (where $\mathbf{s}_1^- = \mathbf{s}_2^+$), then*

$$\begin{aligned} (p_2 - p_1) \cdot \mathbf{s}_1 &< 0, \\ (p_2 - p_1) \cdot \mathbf{s}_2 &< 0, \\ (p_2 - p_1) \cdot (\mathbf{s}_1^+ \times \mathbf{s}_2^-) &> 0 \end{aligned} \tag{21}$$

First suppose that $\mathbf{s}_1$ and $\mathbf{s}_2$ are ultraparallel and consider (20). The inequality defines an infinite pyramid whose sides are defined where the absolute values in (20) arise from multiplication of ± 1.

Corollary 3.3 implies that $\mathsf{A}(\mathbf{s}_1, \mathbf{s}_2)$ consists of all positive linear combinations of

$$\mathbf{s}_1^-, -\mathbf{s}_1^+, -\mathbf{s}_2^-, \mathbf{s}_2^+.$$

Each of these vectors defines one of the four corners of the infinite pyramid. We show this for two vectors, while the other two vectors follow similar reasoning.

Set $p_2 - p_1 = \mathbf{s}_1^-$, and plug this value into both sides of (20). The left-hand side expression, using (2) and (3), is

$$\mathbf{s}_1^- \cdot (\mathbf{s}_1 \times \mathbf{s}_2) = \mathsf{Det}(\mathbf{s}_1^- \mathbf{s}_1 \mathbf{s}_2) = -\mathbf{s}_2 \cdot (\mathbf{s}_1 \times \mathbf{s}_1^-) = \mathbf{s}_2 \cdot \mathbf{s}_1^-.$$

By the definition of consistent orientation, this term is positive. The right-hand side expression is

$$|\mathbf{s}_1^- \cdot \mathbf{s}_1| + |\mathbf{s}_1^- \cdot \mathbf{s}_2| = |\mathbf{s}_1^- \cdot \mathbf{s}_2|.$$

Thus, the vector $p_2 - p_1 = \mathbf{s}_1^-$ defines the ray on the corner with the sides defined by $\mathbf{s}_2 \cdot \mathbf{s}_1^- = |\mathbf{s}_1^- \cdot \mathbf{s}_2|$.

Now, set $p_2 - p_1 = -\mathbf{s}_1^+$, and plug this value into both sides of (20). The left-hand side expression, using (2) and (3), is

$$-\mathbf{s}_1^+ \cdot (\mathbf{s}_1 \times \mathbf{s}_2) = -\mathsf{Det}(\mathbf{s}_1^+\mathbf{s}_1\mathbf{s}_2) = \mathbf{s}_2 \cdot (\mathbf{s}_1 \times \mathbf{s}_1^+) = \mathbf{s}_2 \cdot \mathbf{s}_1^-.$$

By the definition of consistent orientation, this term is positive. The right-hand side expression is

$$| - \mathbf{s}_1^+ \cdot \mathbf{s}_1| + | - \mathbf{s}_1^\pm \cdot \mathbf{s}_2| = |\mathbf{s}_2^- \cdot \mathbf{s}_2|.$$

Thus, the vector $p_2 - p_1 = -\mathbf{s}_1^-$ defines the ray on the corner with the sides defined by $\mathbf{s}_2 \cdot \mathbf{s}_1^- = |\mathbf{s}_1^- \cdot \mathbf{s}_2|$.

The asymptotic case is similar. The set of allowable translations, defined by (21), has three faces whose bounding rays are parallel to

$$\mathbf{s}_2^-,\ -\mathbf{s}_2^+ = -\mathbf{s}_1^-,\ \mathbf{s}_1^+.$$

□

7. Crooked foliations

Now we consider how the previous descriptions of the disjointness of crooked halfspaces impacts the understanding of foliations of E by crooked planes. We provide the following example of a foliation as a preview.

Definition 7.1. Let $0 \le q \le m$. Consider the coordinate projection

$$\mathbb{R}^m \xrightarrow{\Pi} \mathbb{R}^q.$$

Recall that a *foliation* of codimension q of an m-dimensional topological manifold M^m is a decomposition of M into codimension q submanifolds L_x, called *leaves*, (indexed by $x \in M$) together with an atlas of coordinate charts (homeomorphisms)

$$U \xrightarrow{\psi_U} \mathbb{R}^m$$

such that the inverse images $(\psi_U)^{-1}(y)$, for $y \in \mathbb{R}^q$ are the intersections $U \cap L_x$. A *crooked foliation* of an open subset $\Omega \subset \mathsf{E}$ is a foliation of Ω by PL leaves L_x which are intersections of Ω with crooked planes. More generally, if Ω is an open subset such that $\bar{\Omega}$ is a codimension-0 submanifold-with-boundary, we require that $\partial\Omega$ has a coordinate atlas with charts mapping to open subsets of crooked planes.

Start with a foliation of H^2, the projective model of the hyperbolic plane. Consider a one-parameter group of hyperbolic transformations g_t. All of the non-identity elements share an axis $A_g \subset \mathsf{H}^2$. Choose a geodesic ℓ which is perpendicular (in the hyperbolic plane) to the axis A_g. Then the collection $\{g_t(\ell)\}_{t\in\mathbb{R}}$ is a foliation of H^2.

This foliation of H^2 by geodesics gives rise to a foliation of E by crooked planes. Let $\mathcal{C}$ be the crooked plane whose vertex is the origin o and whose axis is Lorentzian-perpendicular to the geodesic ℓ. It is not hard to check that the vector g^0 is parallel (in E) to all of the geodesics $g_t(\ell)$ and it lies in the stem quadrant of every crooked plane defined by the collection of geodesics in the foliation of H^2.

Consider the collection of crooked planes $\{\gamma_t(\mathcal{C})\}_{t\in\mathbb{R}}$, where $\gamma_t = (g_t, tg^0)$. The difference between any two crooked planes in this collection is a multiple of g^0, so there exist disjoint crooked halfspaces bounded by every pair of crooked planes in the collection, that is this collection is a foliation of E.

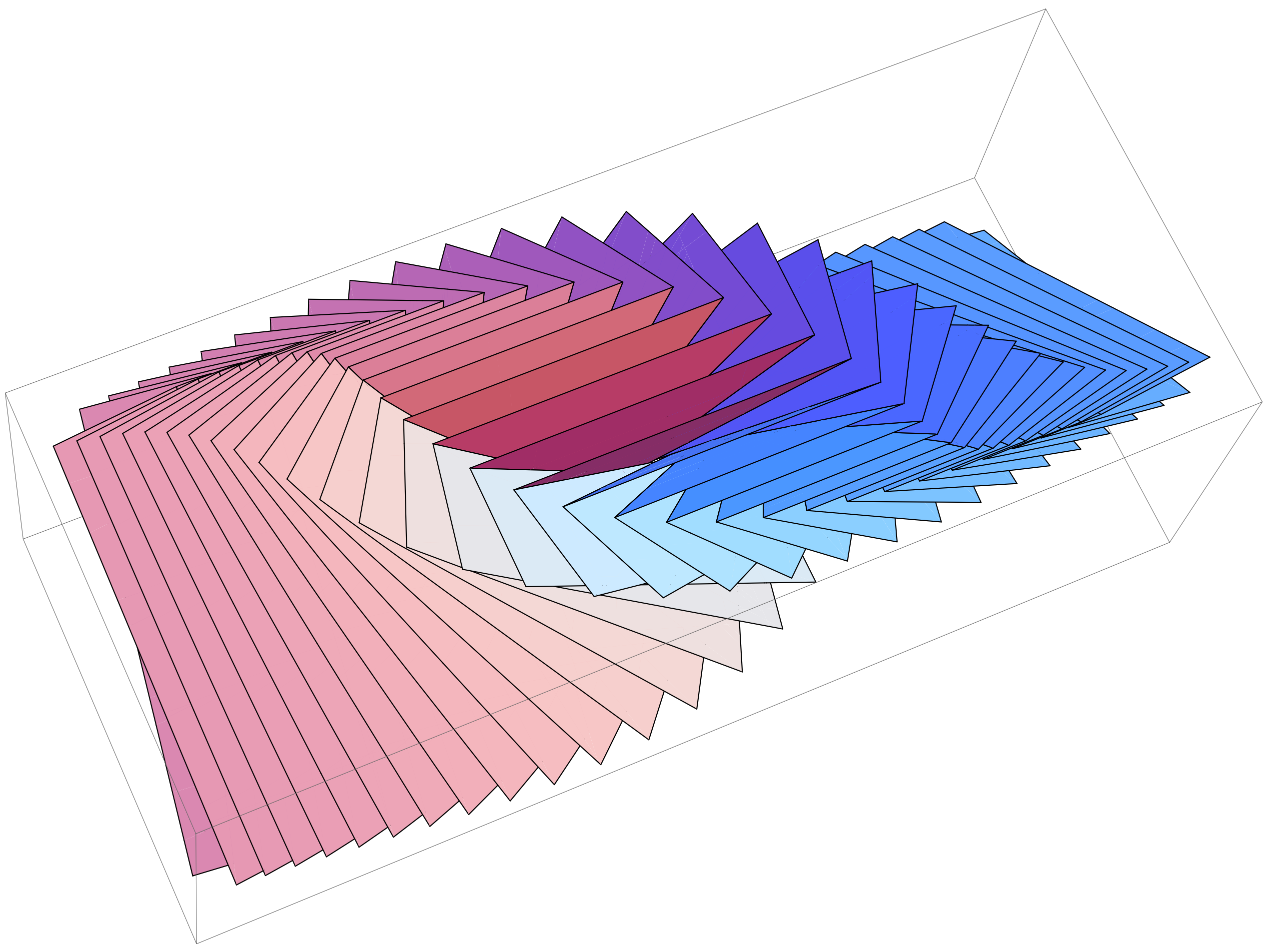

FIGURE 6. A foliation by ultraparallel crooked planes

Now consider foliations of the entire space by crooked planes, and start with a basic example.

Start with a crooked plane $\mathcal{C}$ and consider a one-parameter subgroup of hyperbolic transformations X_t such that their common axis is Lorentzian-perpendicular to the geodesic defined by the spine of $\mathcal{C}$. In the Klein model, suppose without loss of generality that the common axis is the horizontal geodesic through the origin, and the geodesic corresponding to the spine of $\mathcal{C}$ is the vertical geodesic through the origin. The action of X_t yields a foliation of H^2 by Euclidean line segments (in the Klein model) which in turn correspond to spines of crooked planes. Moving to E, a vector with a null third coordinate in the original basis for E must lie in the stem quadrant of every one of the crooked planes $X_t(\mathcal{C})$. Therefore,

$$\left\{X_t(\mathcal{C}) + tX^0\right\}_{t\in\mathbb{R}}$$

foliates the entire affine space E by mutually disjoint crooked planes.

We can in fact prove a slightly more general fact.

Theorem 7.2. *Let $\mathbf{s}_t$, $a \leq t \leq b$ be a continuous path of spacelike-vectors such that $\mathfrak{h}(\mathbf{s}_t)$ foliate a region of H^2 and share a common perpendicular axis. Assume that $\mathcal{H}(\mathbf{s}_t) \subset \mathcal{H}(\mathbf{s}_a)$ for all $a \leq t \leq b$. Let p_t, $a \leq t \leq b$ be a regular path in E such that p'_t belongs to the interior of the the translational semigroup $\mathsf{V}(\mathcal{H}(\mathbf{s}_t))$. Then for every $a \leq t_1, t_2 \leq b$, the crooked planes $\mathcal{C}(\mathbf{s}_{t_1}, p_{t_1})$ and $\mathcal{C}(\mathbf{s}_{t_2}, p_{t_2})$ are disjoint.*

Proof. Say $t_1 < t_2$. Setting $\mathbf{v}_t = p'_t$:

$$p_{t_2} - p_{t_1} = \int_{t_1}^{t_2} \mathbf{v}_t dt$$

Since $\mathbf{s}_t$ is a continuous path, $\mathfrak{h}(\mathbf{s}_{t_1})$ lies between $\mathfrak{h}(\mathbf{s}_a)$ and $\mathfrak{h}(\mathbf{s}_{t_2})$, which lies between $\mathfrak{h}(\mathbf{s}_{t_1})$ and $\mathfrak{h}(\mathbf{s}_b)$. In particular, for every $t \in (t_1, t_2)$:

$$\mathsf{V}(\mathcal{H}(\mathbf{s}_t)) \subset \mathsf{A}(\mathbf{s}_{t_2}, -\mathbf{s}_{t_1})$$

(Note that $\mathbf{s}_{t_1}$, $\mathbf{s}_{t_2}$ are not consistently oriented but $-\mathbf{s}_{t_1}$, $\mathbf{s}_{t_2}$ are.) Thus every $\mathbf{v}_t$ belongs to $\mathsf{A}(\mathbf{s}_{t_2}, -\mathbf{s}_{t_1})$. Since $\mathsf{A}(\mathbf{s}_{t_2}, -\mathbf{s}_{t_1})$ is a cone, it follows that $p_2 - p_1$ belongs to it as well.

$\square$

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